1. Mathematical Induction

We now move to another proof technique which is often useful in proving statements concerning the set of positive integers. The following principle, known as the Principle of Mathematical Induction, can be proved, but it requires a formal definition of “positive integer” which is outside the scope of this course. For us, the set of positive integers, denoted by \( \mathbb{Z}^+ \), consists of the integers 1, 2, 3, \ldots. In other words, \( \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \) (notice that this is an infinite set).

**Theorem 1** (Principle of Mathematical Induction). Suppose that \( S \) is a set of positive integers which has the following two properties:

1. \( 1 \in S \), and
2. for every positive integer \( n \): if \( n \in S \), then also \( n + 1 \in S \).

Then \( S = \mathbb{Z}^+ \), that is, \( S \) contains every positive integer.

As I said earlier, a proof of this statement is beyond the scope of this course. However, I can present a heuristic argument that I hope most of you can find plausible and that I hope you all take the time to read. It is as follows: suppose that \( S \) is a set of positive integers which has properties (1) and (2) above. Then by the first property, 1 is a member of \( S \), that is, \( 1 \in S \). Now, let’s analyze the second property in more detail. It says that for EVERY positive integer \( n \): IF \( n \) happens to be in \( S \), THEN \( n + 1 \) is also in \( S \). We know that \( 1 \in S \). Thus, by property (2), also \( 1 + 1 = 2 \in S \). So we now know that \( 2 \in S \). Applying property (2) again, we see that \( 2 + 1 = 3 \in S \). Now \( 3 \in S \). Applying property (2) again, \( 4 \in S \). Continuing in this manner, it should not be too hard to see that every positive integer is a member of \( S \).

OK, so now that we have stated the abstract principle, how does one use this to prove statements about the positive integer. Let’s start slowly.

**Example 1.** Let \( P(n) \) be the predicate \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \), where the domain for this predicate is the set \( \mathbb{Z}^+ \) of positive integers. (this is important!) Note that \( P(1) \) is \( 1 = \frac{1(1+1)}{2} \), \( P(2) \) is \( 1 + 2 = \frac{2(2+1)}{2} \), \( P(3) \) is \( 1 + 2 + 3 = \frac{3(3+1)}{2} \), etc.

Take a moment and see if you can see what \( P(4) \), \( P(5) \), and \( P(6) \) are. They are, or course, propositions. Are they TRUE propositions? If you are doing things right, you should be able to see that the answer is “yes”. Even more is true: \( P(n) \) is true for EVERY positive integer \( n \). We can use the Principle of Mathematical Induction to prove this fact. Please play close attention to the details of the following proof and try to mimick them in your homework.

**Example 2.** Prove that \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \) for every positive integer \( n \).

**Proof.** Let \( S \) be the set of all positive integers \( n \) for which \( 1 + 2 + 3 + \cdots + n = \frac{n}{n+1} \). We want the above equation to hold for ALL positive integers \( n \), that is, we want our set \( S \) to be equal to the set \( \mathbb{Z}^+ \), the set of all positive integers. Take the time to reread the principle of mathematical induction.
induction. It says that if \( S \) has properties (1) and (2) given in the principle, then \( S \) is, in fact, the set of all positive integers, which is what we want to prove. Thus, by the Principle of Mathematical Induction, it suffices to prove that \( S \) has both of these properties. Proving properties (1) and (2) is called the base case and the inductive step, respectively.

(i) (base case) We must check that \( 1 \in S \), that is, we must check that \( 1 = \frac{1(1+1)}{2} \). This is obvious, since \( \frac{2}{2} = 1 \).

(ii) (inductive step) We must prove that for every positive integer \( n \), IF \( n \in S \), then \( n+1 \in S \). In virtual all proofs by induction, you will prove this using a DIRECT PROOF. So let \( n \) be an arbitrary positive integer. Assume the “if” part, that is, assume that \( n \in S \). We must show the “then” part, that is, \( n+1 \in S \). What does \( n \in S \) mean? It means, by definition, that the equation we are trying to prove for every positive integer is true of \( n \), that is, it means that \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \). We must prove that \( n+1 \in S \), that is, that the equation holds with the \( n \) replaced by \( n+1 \). This means that we are trying to prove that \( 1 + 2 + 3 + \cdots + n + n + 1 = \frac{(n+1)(n+1+1)}{2} \) (take a minute to process that this is exactly the equation we began the problem with, but we replaced every occurrence of \( n \) in the equation with \( n+1 \)). What have assumed is that \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \). How to we get to our destination? Well, first note that the left hand side of the destination equation is the same as our assumed equation, except that it has an additional \( n+1 \) term. So let’s begin by adding \( n+1 \) to both sides of our assumed equation and then try to make the new right hand side the same as the right hand side of our destination equation: we have assumed that \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \). Adding \( n+1 \) to both sides, we get \( 1 + 2 + 3 + \cdots + n + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+1+1)}{2} \), and we have proved the equation we needed to establish. \( \square \)

Now let’s take a look at another similar example.

**Example 3.** Prove that for every positive integer \( n \), we have \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \).

**Proof.** (this will be a bit more streamlined) Let \( S \) be the set of all positive integers \( n \) such that \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \). We again establish both the base case and the inductive step.

(i) (base case) We must check that \( 1 \in S \). This means that the equation above holds when \( n = 1 \), that is, \( 1 = 1^2 \). This is clearly true (you don’t need to wrote more than this, since it is, in fact, obvious).

(ii) (inductive step) Let \( n \) be an arbitrary positive integer, and assume that \( n \in S \). We must prove that \( n+1 \in S \). Since \( n \in S \), by definition of \( S \), we see that (*) \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \). We must prove that \( n+1 \in S \), that is, we must prove that the previous equation holds with \( n \) replaced with \( n+1 \). So we must prove that \( 1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) = (n+1)^2 \). Remember, (*) above is “in our hand” and we must use it to prove the previous equation. So begin by adding \( (2(n+1) - 1) \) to both sides of (*) to get \( 1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) = n^2 + (2(n+1) - 1) \). Now, the right hand side simplifies as follows: \( n^2 + (2(n+1) - 1) = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n+1)^2 \).
Thus we have shown that $1 + 3 + 5 + \cdots + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$, which is what needed to be shown. This completes the proof. \hfill \blacksquare

Our final example (for now) will utilize the Principle of Mathematical Induction to establish an inequality. Throughout the proof, I will be somewhat verbose and remind of some things we’ve discussed previously concerning inequalities.

**Example 4.** Prove that for every real number $x > -1$, we have $(1+x)^n \geq 1+nx$ for every positive integer $n$.

**Proof.** OK, this this one is going to be a bit different/trickier than the previous two examples. Note that the problem begins with, “Prove that for every real number $x > -1$” and then asks you to prove something about every positive integer relative to this $x$. So we start by letting $x$ be an arbitrary real number and we assume that $x > -1$. This sets the stage for the $x$ in the problem. What remains is to prove the assertion about $n$ for every positive integer $n$ by induction. So, like usual, we start by letting $S$ be the set of all positive integers $n$ such that $(1+x)^n \geq 1+nx$. We complete the same two cases as before, but since we are now working with inequalities, the details will be a bit different.

(i) (base case) We must check that $1 \in S$, that is, we must check that $(1+x)^1 \geq 1+1x$, which reduces to $1+x \geq 1+x$, which is obviously true.

(ii) (inductive step) Let $n$ be an arbitrary positive integer. Assume that $n \in S$. We must show that $n+1 \in S$ as well. Since $n \in S$, this means that $(1+x)^n \geq 1+nx$. We must show that $n+1 \in S$, that is, the previous inequality holds with $n$ replaced by $n+1$. So we must prove that
\[(*) \quad (1+x)^{n+1} \geq 1+(n+1)x.\]
The idea here is similar to the idea in the previous two problems. What can we do to both sides of the inequality (1+x)\(^n\) \geq 1+nx in order to change the left side to the left side of the inequality (1+x)\(^{n+1}\) \geq 1+(n+1)x? Can we multiply both sides by 1+x? That will do it. But is this a legal move? Well, remember that x \(>-1\), and so adding 1 to both sides, we get 1+x \(>0\). Since we are multiplying through by a positive value, the inequality sign won’t change (we proved this in earlier homework/notes). So, multiplying both sides of (1+x)\(^n\) \geq 1+nx by the positive 1+x, we get (1+x)\(^n\) \geq 1+(n+1)x. Take a moment to ground yourself; recall that we are trying to prove the inequality (*) above, and right now, we are here: (**) (1+x)\(^{n+1}\) \geq (1+nx)(1+x). If we can simplify the right side to 1+(n+1)x, then we will be done. Let’s see if we can do that: (1+nx)(1+x) = 1+nx+x+nx\(^2\). Now, the right side, 1+(n+1)x is the same as 1+nx+x. So we see that 1+nx+x (our “desired” right side) is not the same as our actual right side of 1+nx+x+nx\(^2\). However, (and this is important), note that x\(^2\) \(\geq 0\) (I proved in class some time ago that the square of any real number is greater than or equal to zero). Recall that n is a positive integer. So we can multiply both sides of x\(^2\) \(\geq 0\) by n to obtain nx\(^2\) \(\geq 0\). Now add 1+nx+x to both sides (see assumption (6)) of nx\(^2\) \(\geq 0\) to get (***) 1+nx+x+nx\(^2\) > 1+nx+x.

Now let’s take stock of everything we have done up to this point. Recall that we had (1+x)\(^{n+1}\) \geq (1+nx)(1+x). After multiplying out the right side, this is the same as (1+x)\(^{n+1}\) \geq 1+nx+x+nx\(^2\). Now, from (***) above, we have 1+nx+x+nx\(^2\) > 1+nx+x. So using the fact that if a, b, c are real
numbers and \( a \geq b \) and \( b > 0 \), then \( a > c \) (this is an easy proof by cases), we get \((1+x)^{n+1} > 1+nx+x\). Factoring the right side, this yields \((1+x)^{n+1} > 1+(n+1)x\), and hence \((1+x)^{n+1} \geq 1+(n+1)x\), as was to be shown. \( \square \)

We will now close out this section by introducing a slightly more general form of the Principle of Mathematical Induction.

**Theorem 2** (General Principle of Mathematical Induction). Let \( m \) be a fixed integer. Suppose that \( S \) is a set of integers, all at least \( m \) (this is important) which satisfies the following conditions:

1. \( m \in S \), and
2. for all integers \( n \geq m \): if \( n \in S \), then \( n+1 \in S \).

Then \( S \) is equal to the set of all integers at least \( m \); that is, \( S = \{m, m+1, m+2, \ldots \} \).

The idea behind this is similar to the idea behind the first principle introduced. First, \( m \in S \) by (1). Now by (2), since \( m \in S \), also \( m+1 \in S \). Applying (2) again, since \( m+1 \in S \), also \( m+2 \in S \). Applying (2) again, since \( m+2 \in S \), also \( m+3 \in S \). Continuing this process, we see that every integer at least \( m \) is a member of \( S \).

Now let’s look at an example using this more general principle of mathematical induction.

**Example 5.** Suppose that \( r \) is a real number not equal to 1. Then for every integer \( n \geq 0 \), we have \( r^0 + r^1 + \cdots + r^n = \frac{r^{n+1} - 1}{r-1} \). \( \square \)

Proof. Let \( r \) be an arbitrary real number not equal to 1. Now let \( S \) be the set of all integers \( n \geq 0 \) which satisfy the above equation. We must show that \( S \) is the set of all nonnegative integers. Note that we are now proving a statement for all non-negative integers, NOT for all positive integers. So the base case changes to 0, not to 1. IN GENERAL, THE BASE CASE WILL CONSIST OF CHECKING THE STATEMENT YOU’RE TRYING TO PROVE FOR THE SMALLEST NUMBER IN THE SET OF INTEGERS UNDER CONSIDERATION.

(i) (base case) We must check that \( 0 \in S \), that is, we must check that \( r^0 = \frac{r^{0+1} - 1}{r-1} \). Note that this equation simplifies to \( 1 = \frac{r-1}{r-1} \), which is clearly true. Thus \( 0 \in S \).

(ii) (inductive step) Let \( n \) be an arbitrary non-negative integer (NOTE THAT WE HAVE NOT STATED THAT \( n \) IS AN ARBITRARY POSITIVE INTEGER. IN THIS STEP, YOU LET \( n \) BE AN ARBITRARY INTEGER IN THE SET OF INTEGERS UNDER CONSIDERATION. NOW THE RELEVANT SET OF INTEGERS IS THE SET OF NON-NEGATIVE INTEGERS). Assume that \( n \in S \). We must show that \( n+1 \in S \). Since \( n \in S \), this means that \( r^0 + r^1 + \cdots + r^n = \frac{r^{n+1} - 1}{r-1} \). We must prove that \( n+1 \in S \), that is, we must show that \( r^0 + r^1 + \cdots + r^n + r^{n+1} = \frac{r^{n+2} - 1}{r-1} \). The equation we have in our hand is (*): \( r^0 + r^1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r-1} \). Toward this end, add \( r^{n+1} \) to both sides of (*) to get \( r^0 + r^1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r-1} + r^{n+1} = \frac{r^{n+1} - 1}{r-1} + \frac{r^{n+2} - r^{n+1}}{r-1} = \frac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r-1} = \frac{r^{n+2} - 1}{r-1} \), as was to be shown. \( \square \)